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Magnetic Field Controlled Buckling and Post-buckling of an Inclined Beam

Assuming that the 2x2x2 Octet-truss lattice undergoes homogeneous deformation, the characteristic unit is a rubber beam extracted from the lattice. We model the rubber beam as an elastica of bending stiffness EI and length L. To make the model clear, **Fig.1** is given to help the visualization. In the figure, the left end of the elastica is fixed on the base with a constant inclination angle α, and the right end of the elastica is fixed on a roller, which has a vertical translational degree of freedom.

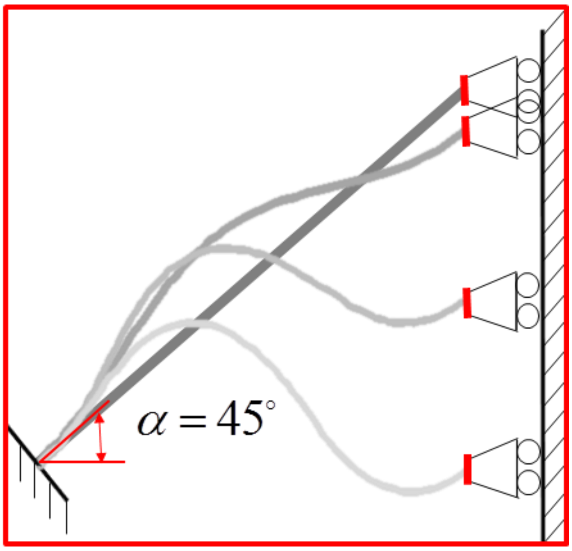

Fig.1

Subject to no loadings, the elastica can retain its naturally straight configuration. After applying a uniform magnetic induction field $\mathbf{B}$, how will the elastica behave? In this paper, we attempt to address this problem and figure out an approach to predict the value of the vertical displacement of the right end of the elastica, or the roller.

To begin with, recognizing that there exist coupling between elasticity and magnetization, we will analyze the problem by energy method. The system, which is composed of the elastica, the magnetic loading and the boundary conditions, comprises the elastic energy, applied magnetic energy and demagnetization energy. Fig. 2 is given to help setting up the concepts and notations to describe the problem.

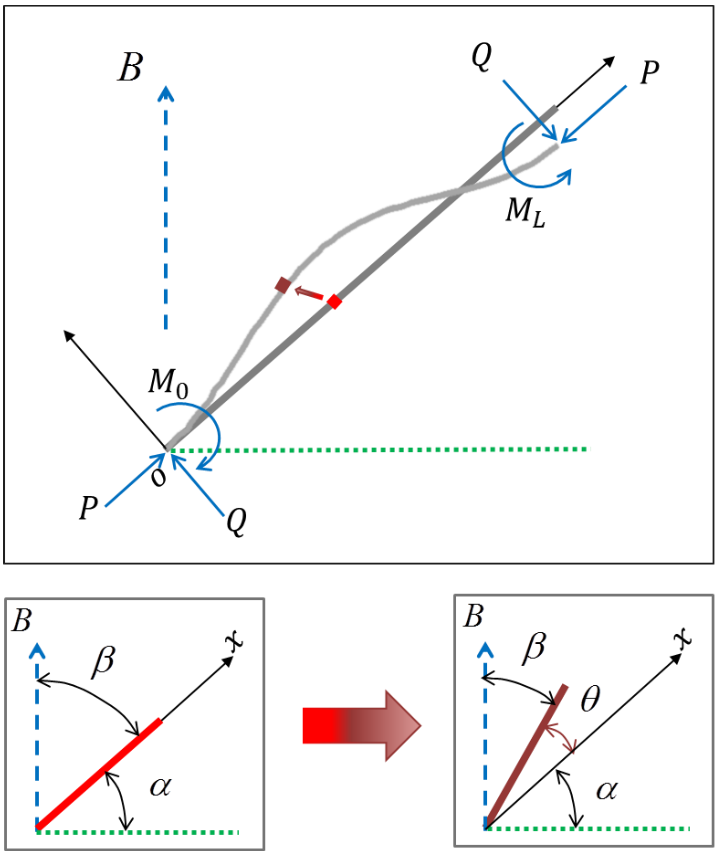

Fig.2

Only the bending energy remains to be accounted for. For the large deflection bending of an elastica, the midline through the centroid of cross-section is our study object. At its initial configuration Ω, the midline is straight. Its natural arc length s is used to differentiate each material particle. A Cartesian coordinate system is established to capture the position of each material particle.

$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \hat{x}(s) \\ \hat{y}(s) \end{pmatrix} \tag{1}$$

The deformation gradient is denotes as

$$\mathbf{F} = \frac{d\mathbf{r}}{ds} = \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix} \tag{2}$$

The local expansion of a material fiber in the beam is denoted as

$$J = \sqrt{\det(\mathbf{F}^T\mathbf{F})} \tag{3}$$

Since the material is deemed as inextensible, the constraint $J = 1$ holds. In addition, the unit tangent at current configuration is

$$\mathbf{t} = \frac{1}{J}\mathbf{F} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{4}$$

where, θ is the angle between $\mathbf{t}$ and its initial direction, we call it **rotation angle**. Rewriting Eq.(2) in terms of Eq.(4) gives

$$\frac{dx}{ds} = \cos\theta \tag{5}$$

$$\frac{dy}{ds} = \sin\theta \tag{6}$$

By virtue of Frenet formula, one arrives at

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} \tag{7}$$

where $\mathbf{n}\cdot\mathbf{t} = 0$ and $\mathbf{n}$ is the unit normal defined as

$$\mathbf{n} = \frac{d\mathbf{t}}{d\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{8}$$

Thus, the curvature can be expressed as a function of the deformation gradient.

$$\kappa = \frac{d}{ds}\left(\frac{1}{J}\mathbf{F}\right)\cdot\mathbf{n} = \frac{d\mathbf{t}}{d\theta}\cdot\mathbf{n}\frac{d\theta}{ds} = \frac{d\theta}{ds} \tag{9}$$

The bending energy can be expressed as

$$E_b = \int_0^L \frac{EI}{2}\left(\frac{d\theta}{ds}\right)^2 ds \tag{10}$$

Where E is Young's modulus and I moment of cross-section area.

Besides of the elastic energy, the magnetic enthalpy also contributes to the total free energy. To study the magnetic enthalpy, the elastica is equivalently deemed to composed of a series of isotropic magneto-rigid ellipsoids. Their principal radius vectors are denoted as $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, where the following relationship is assumed.

$$\mathbf{r}_1 = r_1\mathbf{t} \tag{11}$$

$$\mathbf{r}_2 = r_2\mathbf{n} = r\mathbf{n} \tag{12}$$

$$r_1 \geq r_2 = r_3 = r \tag{13}$$

As the axial symmetry, we consider a two dimensional plane expanded by $\mathbf{r}_1$ and $\mathbf{r}_2$. Let the ellipsoid be subject to a uniform magnetic induction field

$$\mathbf{B} = \mu_0\mathbf{H}_0 \tag{14}$$

where the angle between $\mathbf{r}_1$ and $\mathbf{B}$ is denoted as β.

The magnetic induction field exerts a moment to the ellipsoid. A constant magnetization $\mathbf{M}$ will form inside the ellipsoid and depend on the internal magnetic field $\mathbf{H}$.

$$\mathbf{M} = \chi(H)\mathbf{H} \tag{15}$$

where the angle between $\mathbf{r}_1$ and $\mathbf{H}$ is denoted as γ.

The internal magnetic field can be determined based on the shape of the ellipsoid and the demagnetizing factors

$$\mathbf{H} = \mathbf{H}_0 - \bar{\mathbf{n}} \cdot \mathbf{M} \tag{16}$$

where the demagnetizing factor

$$\mathbf{n} = \begin{bmatrix} n_1 & 0 \\ 0 & n_2 \end{bmatrix} \tag{17}$$

Then, one can obtain

$$\mathbf{M} = \chi_{eff}\mathbf{H}_0 = H_0\begin{pmatrix} \chi/(1+n_1\chi)\cos\beta \\ \chi/(1+n_2\chi)\sin\beta \end{pmatrix} = H_0\begin{pmatrix} \chi_1\cos\beta \\ \chi_2\sin\beta \end{pmatrix} \tag{18}$$

By using the independent model [] for magnetic enthalpy, the free energy of a magnetic elastica is expressed as

$$\Pi\left(\theta\right)=\int_0^L \frac{EI}{2}\left(\frac{d\theta}{ds}\right)^2 ds-\int_0^L A\int_0^{\mathbf{B}_0}\mathbf{M}\cdot d\mathbf{B}ds \tag{19}$$

Noting that $\beta=\pi/2-\theta-\alpha$, after substituting Eq.(14) and Eq.(18) to Eq.(19), one arrives at

$$\begin{aligned}\Pi\left(\theta\right)&=\int_0^L \frac{EI}{2}\left(\frac{d\theta}{ds}\right)^2 ds-\int_0^L A\mu_0\int_0^{H_0}\bar{H}_0\chi_2\cos^2\left(\theta+\alpha\right)+\chi_1\sin^2\left(\theta+\alpha\right)d\bar{H}_0 ds\\&=\int_0^L \frac{EI}{2}\left(\frac{d\theta}{ds}\right)^2 ds-\int_0^L \frac{A\mu_0H_0^2}{2}\left[\chi_2\cos^2\left(\theta+\alpha\right)+\chi_1\sin^2\left(\theta+\alpha\right)\right]ds\end{aligned} \tag{20}$$

The

$$\begin{aligned}\Pi\left(\theta+\varepsilon\eta\right)=\Pi\left(\theta\right)+\Delta\Pi\left(\eta\right)&=\int_0^L \frac{EI}{2}\left(\frac{d\theta}{ds}+\varepsilon\frac{d\eta}{ds}\right)^2 ds\\&-\int_0^L A\mu_0\int_0^{H_0}\bar{H}_0\chi_2\cos^2\left(\theta+\varepsilon\eta+\alpha\right)+\chi_1\sin^2\left(\theta+\varepsilon\eta+\alpha\right)d\bar{H}_0 ds\\&=\int_0^L \frac{EI}{2}\left(\frac{d\theta}{ds}+\varepsilon\frac{d\eta}{ds}\right)^2 ds\\&-\int_0^L \frac{A\mu_0H_0^2}{2}\left[\chi_2\cos^2\left(\theta+\varepsilon\eta+\alpha\right)+\chi_1\sin^2\left(\theta+\varepsilon\eta+\alpha\right)\right]ds\end{aligned} \tag{21}$$

The stationary condition requires that

$$\Delta\Pi\left(\eta\right)=\delta\Pi+\frac{1}{2}\delta^2\Pi+O(s^3) \tag{22}$$

The first order variation of the free energy is

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\tilde{\Pi}\left(\theta+\varepsilon\eta\right)=\delta\Pi=\int_0^L EI\left(\frac{d\theta}{ds}\right)\frac{d\eta}{ds}ds-\int_0^L \frac{A\mu_0H_0^2}{2}\left(\chi_1-\chi_2\right)\sin\left(2\theta+2\alpha\right)\eta ds \tag{23}$$

By move the derivative, one obtains

$$\delta\Pi=\int_0^L\left[-EI\left(\frac{d^2\theta}{ds^2}\right)-\frac{A\mu_0H_0^2}{2}\left(\chi_1-\chi_2\right)\sin\left(2\theta+2\alpha\right)\right]\eta ds \tag{24}$$

The equation of local equilibrium is

$$EI\frac{d^2\theta}{ds^2}+\frac{A\mu_0H_0^2\Delta\chi}{2}\sin\left(2\theta+2\alpha\right)=0 \tag{25}$$

where $\Delta\chi=\chi_1-\chi_2=\chi/\left(1+n_1\chi\right)-\chi/\left(1+n_2\chi\right)$. Integrate Eq.(25) once with respect to s, one can obtain

$$EI\frac{d\theta}{ds}=-\int_0^s \frac{A\mu_0H_0^2\Delta\chi}{2}\sin\left(2\theta+2\alpha\right)ds=M_B\left(s\right) \tag{26}$$

We can see that the magnetic fields exerts a distributed moment to the elastic. The moment is denoted as $M_B(s)$.

Through further derivation[], one obtains the second order variation of the free energy

$$
\begin{aligned}
\frac{1}{2}\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0}\Pi(\theta+\varepsilon\eta) &= \frac{1}{2}\delta^2\Pi = Q(\eta) \\
&= \int_0^L \frac{EI}{2}\left(\frac{d\eta}{ds}\right)^2 ds - \int_0^L \frac{A\mu_0 H_0^2}{2}\tilde{\chi}\eta^2 ds \\
&= \frac{EI}{2}\left(\frac{d\eta}{ds}\right)\eta\bigg|_0^L - \int_0^L \frac{EI}{2}\frac{d^2\eta}{ds^2}\eta ds - \int_0^L \frac{A\mu_0 H_0^2}{2}\tilde{\chi}\eta^2 ds
\end{aligned}
\tag{27}
$$

where $\tilde{\chi} = (\chi_1 - \chi_2)\cos(2\theta + 2\alpha) = \Delta\chi\cos(2\theta + 2\alpha)$.

Note that $\tilde{\chi}$ is a constant in terms of η because θ is unchanged. Besides, note that this term is distinctly different from the one in the free energy. Using the neutral equilibrium criterion, one can get the buckling equation and its boundary condition.

$$
\begin{cases}
d^2\eta/ds^2 + \lambda^2\eta = 0 \\
\eta|_{s=0} = 0 \\
\eta|_{s=L} = 0
\end{cases}
\tag{28}
$$

where $\lambda^2 = \frac{A\mu_0 H_0^2}{EI}\tilde{\chi} = \frac{AB^2}{\mu_0 EI}\tilde{\chi}$

The general solution for Eq.(28) is

$$\eta(s) = C_1\sin(\lambda s) + C_2\cos(\lambda s) \tag{29}$$

After plugging the boundary conditions, one arrives at

$$\sin(\lambda L) = 0 \tag{30}$$

which means the critical value for λ is the smallest one, which satisfies Eq. (30)

$$\lambda_{cr} = \frac{\pi}{L} \tag{31}$$

Finally, from Eq.(31) and the definition of λ, the critical magnetic induction field can be obtained as

$$B_{cr} = \frac{\pi}{L}\sqrt{\frac{I}{A}}\sqrt{\frac{\mu_0 E}{\tilde{\chi}}} \tag{32}$$

After plugging $I = \pi r^4/4$ and $A = \pi r^2$ into Eq.(32), the critical magnetic induction field can be determined as

$$B_{cr} = \frac{\pi}{2}\frac{r}{L}\sqrt{\frac{\mu_0 E}{\tilde{\chi}}} \tag{33}$$

From Eq.(33), we know the critical magnetic induction depends linearly on the aspect ratio and the generalized stiffness respectively. **Fig**.3

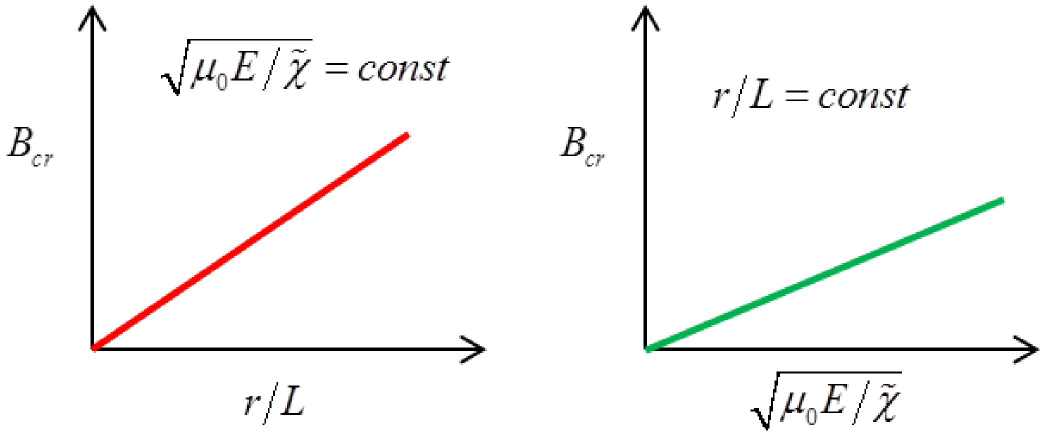

Fig.3

After buckling, we can further analyze the post-buckling behavior of the elastica. For the elastica in a slightly deflected configuration, see **Fig.2.** Because of the vertical movement of the roller at the right end of the elastica, the reaction force will exert extra moments to the elastica. Denote the vertical displacement of the roller as δ. We can get the coordinate of the roller as

$$\begin{aligned} x_r &= L - \delta \sin\alpha \\ y_r &= -\delta \cos\alpha \end{aligned} \tag{34}$$

For any differential element in the elastica, the moment can be expressed as

$$M(s) = -P(y - y_r) - Q(x_r - x) + M_B(s) + M_L \tag{35}$$

Note that the reaction force with amplitude R is normal to the right boundary surface, P and Q can be expressed as

$$P = R\cos\alpha \tag{36}$$

$$Q = -R\sin\alpha \tag{37}$$

Thus, Eq.(35) can be reduced as

$$M(s) = -R\cos\alpha(y - y_r) + R\sin\alpha(x_r - x) + M_B(s) + M_L \tag{38}$$

By differentiating the above equation with respect to s and using the Euler-Bernoulli moment curvature relationship, one will obtain the post-buckling governing equation, which is different from the pre-buckling governing equation Eq. (25).

$$EI\frac{d^2\theta}{ds^2} = -R\cos\alpha\sin\theta - R\sin\alpha\cos\theta - \frac{\pi r^2 B^2 \Delta\chi}{2\mu_0}\sin(2\theta + 2\alpha) \tag{39}$$

After integrating once with respective to θ, one gets

$$\frac{EI}{2}\left(\frac{d\theta}{ds}\right)^2 = R\cos(\alpha + \theta) + \frac{\pi r^2 B^2 \Delta\chi}{4\mu_0}\cos(2\theta + 2\alpha) + Const \tag{40}$$

Rearranging Eq.(40), one can explicitly obtain the expression for the curvature of the beam at any point:

$$\frac{d\theta}{ds} = \pm\sqrt{\left(2R\cos(\alpha + \theta) + \frac{\pi r^2 B^2 \Delta\chi}{2\mu_0}\cos(2\theta + 2\alpha) + 2Const\right)\Big/ EI} \tag{41}$$

Setting:

$$2R/EI = D \tag{42}$$

$$\frac{\pi r^2 B^2 \Delta\chi}{2\mu_0 EI} = T \tag{43}$$

$$2Const/EI = C \tag{44}$$

Hereto, the curvature Eq.(41) can be expressed as

$$\frac{d\theta}{ds}=\pm\sqrt{D\cos(\theta+\alpha)+T\cos(2\theta+2\alpha)+C} \tag{45}$$

From Eq.(45), the rotation angle of the inflection point θ^* can be determined by the following equation

$$C=-D\cos(\theta^*+\alpha)-T\cos(2\theta^*+2\alpha) \tag{46}$$

From the expression above, we know that C depends on the loading parameters and rotation angles of the inflection points.

And the moment

$$M(\theta)=\pm EI\sqrt{D\cos(\theta+\alpha)+T\cos(2\theta+2\alpha)+C} \tag{47}$$

By using the boundary condition

$$\begin{aligned}\theta\big|_{s=0}&=0\\ \theta\big|_{s=L}&=0\end{aligned} \tag{48}$$

and plugging them into Eq.(47), one can get

$$\begin{aligned}C&=\left(\frac{M_L+M_B(L)}{EI}\right)^2-D\cos(\alpha)-T\cos(2\alpha)\\ &=\left(\frac{M_0+M_B(0)}{EI}\right)^2-D\cos(\alpha)-T\cos(2\alpha)\end{aligned} \tag{49}$$

For the odd mode, knowing that $M_B(0)=0$, one has the condition

$$M_0=-M_L-M_B(L) \tag{50}$$

From Eq.(38) and, one can get

$$M_L=\frac{-R\cos\alpha y_r-R\sin\alpha x_r}{2}-M_B(L) \tag{51}$$

If plugging Eq.(46) into Eq.(49), one can get another relationship between M_L and θ^*, as following

$$\left(\frac{M_L + M_B(L)}{EI}\right)^2 = D\left[\cos(\alpha) - \cos(\theta^* + \alpha)\right] + T\left[\cos(2\alpha) + \cos(2\theta^* + 2\alpha)\right] \quad (52)$$

By combining Eq.(51) and Eq.(52), one is able to eliminate the M_L

For the even mode, one has the condition

$$M_0 = M_L + M_B(L) \quad (53)$$

By using Eq.(38), one get.

$$R\cos\alpha y_r + R\sin\alpha x_r = M_B(L) \quad (54)$$

The moment can be determined by Eq.(52) as

$$M_L = EI\sqrt{D\left[\cos(\alpha) - \cos(\theta^* + \alpha)\right] + T\left[\cos(2\alpha) + \cos(2\theta^* + 2\alpha)\right]} - M_B(L) \quad (55)$$

To state the problem more clearly, the forthcoming paragraph will give a discussion in detail. Firstly, it is known that the control parameter in the problem is the vertical displacement of the roller δ, i.e. x_r and y_r. What we are concerned about is the unknowns in the problem. Firstly, the reaction forces, which is described by D and M_L, are undetermined. Secondly, the magnetic force, which is described by T, is also required to be solved. Up to now, the condition that we have used is one of the boundary conditions, i.e. Eq.(48), which restrains the rotation angle of both ends of the elastica. From that boundary condition, the moment at the right end of the elastica M_L can be determined as Eq.(51) or Eq.(55) respectively in different cases. Thus, there remain two unknowns D and T here, it is needed to add two constrains to this statically indeterminate problem. Let's put the Eq.(5), Eq.(6) and Eq.(39) which govern the x, y and θ here again to make the derivation clearer

$$\begin{cases} \dfrac{dx}{ds} = \cos\theta \\ \dfrac{dy}{ds} = \sin\theta \\ \dfrac{d^2\theta}{ds^2} = -\dfrac{R}{EI}\sin(\theta+\alpha) - \dfrac{\pi r^2 B^2 \Delta\chi}{2\mu_0 EI}\sin(2\theta+2\alpha) \end{cases} \tag{56}$$

From Eq. (56) and by virtue of Eq.(45), one can easily get the following three equations based on the number n of inflection points.

$$x_r = \int_0^{x_r} dx = \int_0^L \cos\theta ds = \sum_{k=1}^{2n} \int_0^{(-1)^{\langle (k-1)/2 \rangle}\theta^*} \frac{(-1)^{\langle (k-1)/2 \rangle + 1}\cos\theta}{\sqrt{D\cos(\theta+\alpha) + T\cos(2\theta+2\alpha) + C}} d\theta \tag{57}$$

$$y_r = \int_0^{y_r} dy = \int_0^L \sin\theta ds = \sum_{k=1}^{2n} \int_0^{(-1)^{\langle (k-1)/2 \rangle}\theta^*} \frac{(-1)^{\langle (k-1)/2 \rangle + 1}\sin\theta}{\sqrt{D\cos(\theta+\alpha) + T\cos(2\theta+2\alpha) + C}} d\theta \tag{58}$$

$$L = \int_0^L ds = \sum_{k=1}^{2n} \int_0^{(-1)^{\langle (k-1)/2 \rangle}\theta^*} \frac{(-1)^{\langle (k-1)/2 \rangle + 1}}{\sqrt{D\cos(\theta+\alpha) + T\cos(2\theta+2\alpha) + C}} d\theta \tag{59}$$

where the operator $\langle * \rangle$ means a truncation to a number $*$. Thus C, D and T can be determined by Eq. (57)~(59). And θ^*, which is determined by C, D and T , can be determined by Eq.(46).

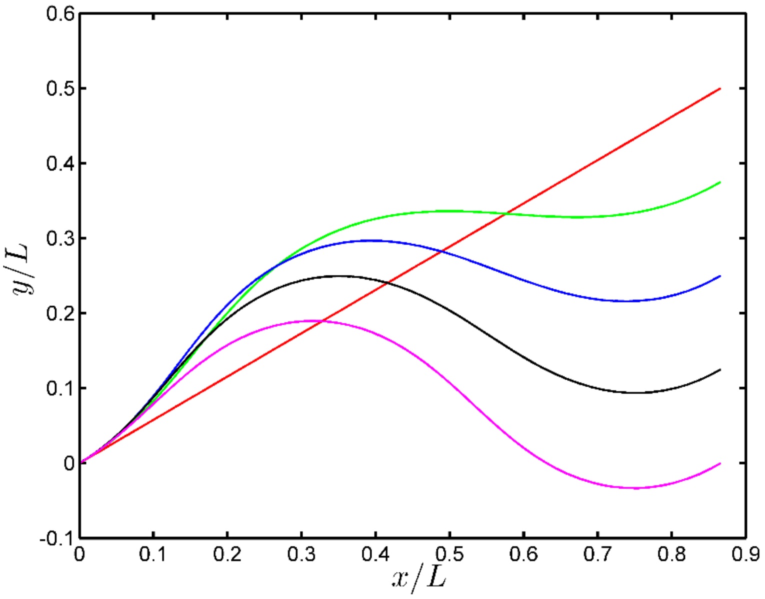

Fig. 4